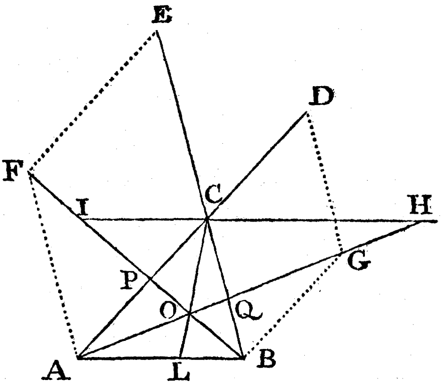
III. *Propoſitions ſelected from a Paper on the Diviſion of Right Lines, Surfaces, and Solids.* *By* James Glenie, *A. M. of the Univerſity of* Edinburgh. *Communicated by the* Aſtronomer Royal.

R. June, 1775.

PROPOSITION I. THEOREM.

If from the angles at the baſe of any right-lined triangle, right lines be drawn to the alternate angles of rhombi, *deſcribed upon the oppoſite ſides, and applied reciprocally to the ſides produced; and from the vertex, through the interſection of theſe lines, a right line be drawn to meet the baſe: the ſegments of the baſe, made thereby, will have to each other the duplicate proportion of the ſides.*



LET ACB be any right lined triangle. Let AFEC, CDGB be rhombi, on any two ſides AC, CB of this triangle, applied reſpectively to CB, AC, produced: from the alternate angles EFA, DGB, of which let FA, GA, be right lines drawn to the angles at the baſe, or third ſide, AB. Then, if through the interſection O of

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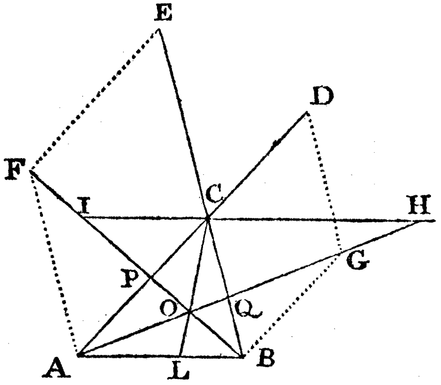


theſe lines, a right line COL be drawn from the vertex C to meet the baſe AB; the ſegments AL, LB, of the baſe made thereby, will have to each other the duplicate proportion of the ſides AC, CB. For through the vertex C let there be a right line drawn parallel to AB, to meet BF, AG produced, if neceſſary. Then, ſince the triangles CQH, CPI, are reſpectively equiangular to the triangles AQB, APB (15. and 29. E. 1.); the proportions of CH to AB and of AB to IC are reſpectively equal to the proportions of CQ to QB and of AP to PC (4. E. 6.). But the proportion of CH to IC is compounded of the proportion of CH to AB, and of AB to IC; and conſequently is equal to the proportion compounded of the proportions of CQ to QB, and of AP to PC. And, ſince the triangles ACQ, APF, are reſpectively equiangular to the triangles BQG, BPC (15. and 29. E. 1.); the proportions of CQ to QB and of AP to PC are each equal to the proportion of AC to CB (4. E. 6.); and when compounded are equal to the duplicate proportion of AC to CB. Wherefore the proportion of CH to IC, which hath been ſhewn to be equal to the proportion compounded of the proportions of CQ to QB and of AP to PC, is alſo equal to the duplicate proportion of AC to CB (11. E. 5.). But, ſince the triangles COH, COI, are reſpectively equiangular to the triangles AOL, LOB, the proportion of CH to IC is equal to the proportion of AL to LB (4. E. 6. and 16. E. 5.). Therefore the proportion of AL to LB is equal to the duplicate proportion of AC to CB (11. E. 5.). *Q. E. D.*

COR. I. If the triangle be iſoſceles, the right line drawn from the vertex to the baſe is perpendicular thereto, and the ſegments of the baſe are equal to each other.

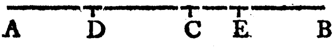
COR. II. When the triangle is right-angled, the line drawn from the vertex to the baſe is always perpendicular to it (as appears from 8. E. 6. and its cor.); and the *rhombi* become ſquares on the ſides comprehending the right angle.

COR. III. The ſegments of the ſides adjacent to the baſe, are reſpectively third proportionals to the ſum of the ſides, and the ſides themſelves.

COR. IV. The ſegments of the ſides adjacent to the vertex are equal to each other, and each of them is a fourth proportional to the ſum of the ſides, and the ſides themſelves(a).

COR.

(a) And it may be added, a mean in proportion between the two ſegments adjacent to the baſe. For if a right line AB be any how divided in C, and from the two ſegments CA, CB, third proportionals to the whole line and each ſegment reſpectively, CD, CE, be taken away, the remainders AD, EB, are equal, and each is a mean in proportion between the two CD, CE. For becauſe AB : AC = AC : CD; therefore,

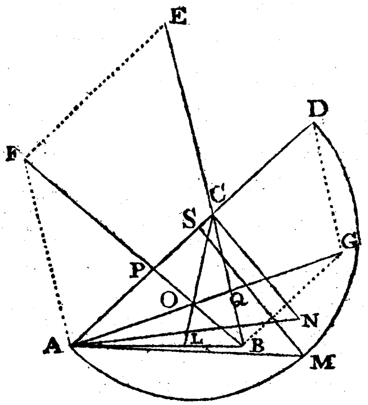


COR. V. The ſegments of the baſe are proportional to the ſegments of the ſides, which are adjacent to them.

PROPOSITION II.

Let there be any two right lines given. There is an angle which may be made by theſe lines; ſuch that if, from their extremities which do not meet, right lines be drawn to the alternate angles of rhombi deſcribed on them, and reciprocally applied to them when produced; and from the ſaid angle through the interſection of theſe lines, a right line be drawn to meet the right line joining the ſaid extremities; the ſegments of this line made thereby, ſhall be reſpectively equal to the adjacent ſegments of the given lines.

therefore, by converſion, $AB : BC = CA : AD$. Again, becauſe $AB : BC = BC : CE$, by converſion $AB : AC = BC : BE$: and by permutation $AB : BC = AC : BE$. Therefore $AC : BE = AC : AD$. Therefore AD and BE are equal. I ſay, moreover, that each of the two equal lines AD, BE, is a mean in proportion between the two CD, CE. For becauſe $BA : AC = AC : CD$, by diviſion $BC : CA = AD : DC$. Again, becauſe $BA : BC = BC : CE$, converting and dividing $BC : CA = CE : EB$. Therefore, $CE : EB = AD : DC$. *Q. E. D.* S. HORSLEY.

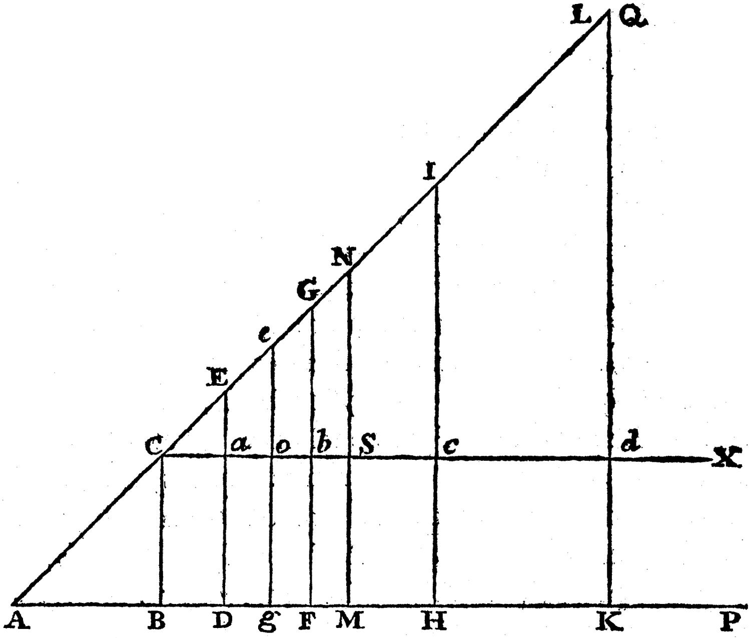


LET AC, CB, be any two given right lines, and let CD in AC produced be equal to CB. On AD deſcribe a ſemi-circle; draw CN at right angles to AD, and equal to CD (11. E. 1.); join A, N, and apply a right line AM in the ſemi-circle equal to AN (1. E. 4.). From the point M draw the right line MS at right angles to AD (1. E. 4.) Make a triangle ACB, having its ſides equal to AC, AS, and CB (by 22. E. 1.); and ACB is the angle required to be found; and the ſegments AL, LB, of the right line AB joining the extremities A and B of the given lines are reſpectively equal to the ſegments AP, BQ, of the given lines, which are adjacent to them. For the ſquare on BC hath to the ſquare on AC the duplicate proportion of BC to AC (cor. 1. to 20. E. 6.); that is, the proportion of BL to LA (prop. 1.). Wherefore the ſquares on AC, CB; that is, the ſquare on AN or AM (47. E. 1.) hath to the ſquare on AC the proportion of AB to AL (18. E. 5.). But the ſquare on AN or AM is equal to the rectangle contained by AD, AS (8. and 17. E. 6.). Wherefore the rectangle contained by AD, AS, hath to the ſquare on AC, the proportion of AB to AL; that is, the proportion of the rectangle contained by AB, AS to the rectangle contained by AL, AS (1. E. 6.). Conſequently, the proportion of the rectangle AD, AS, to the rectangle

rectangle AB, AS; that is, the proportion of AD to AB (1. E. 6.) is equal to the proportion of the ſquare on AC to the rectangle AL, AS (16. E. 5.) But AD hath to AC the proportion of AC to AP (cor. 3. to prop. 1.). Therefore, the rectangle AD, AC, hath to the ſquare on AC the proportion of the rectangle AC, AS, to the rectangle AP, AS (1. E. 6.). Therefore AD hath to AS or AB the proportion of the ſquare on AC to the rectangle AP, AS (16. E. 5. and 1. E. 6.). Hence the rectangle AL, AS, is equal to the rectangle AP, AS (11. and 16. E. 5.), and AL, AP, are conſequently equal. But as AL to AP, ſo is BL to BQ (by cor. 5. prop. 1.). Therefore BL, BQ, are likewiſe equal. *Q. E. D.*

PROPOSITION III. PROBLEM I.

To multiply the ſquare of a given finite right line by any number.



UPON an indefinite right line AP ſet off the given right line AB; draw BC at right angles to AP and equal to AB; and from A through C draw an indefinite right line AQ. Take AD equal to AC, and draw DE parallel to BC; AF equal to AE; and draw FG parallel to BC, and ſo on. Then it appears (from 47. E. 1.), that the ſquare of AC is is equal to the ſquare of AB multiplied by 2; the ſquare of AE equal to the ſquare of AD or AC multiplied by 2; that is, equal to the ſquare of AB multiplied by 4, and ſo on. Thus the ſquares of AC, AE, AG, AI, AL, &c. are reſpectively equal to the ſquare of AB multiplied by the terms of the following ſeries 2, 4, 8, 16, 32, 64, &c. where the ſixty-third term gives the ſquare of AB multiplied by the laſt term of SESSA's Series for the Cheſsboard.

If CX be drawn parallel to AP, the ſquares of A*a*, A*b*, A*c*, A*d*, &c. will be reſpectively equal to the ſquare of AB multiplied by 3, 5, 9, 17, 33, 65, 129, &c. Alſo if A*g* be taken equal to A*a*, and *ge* be drawn parallel to BC, and this be repeated, the ſquares of A*e*, &c. will be equal reſpectively to the ſquare of AB multiplied by 6, 12, 24, 48, &c. And the ſquares on A*o*, &c. will be equal to ſquare on AB multiplied by 4, 7, 13, 25, 49, &c. In like manner, if AM be taken equal to A*b*, and MN be drawn parallel to BC, the ſquares on AN, &c. will be equal reſpectively to the ſquare on AB multiplied by 10, 20, 40, 80, 160, &c. And the ſquares on AS, &c. will be equal reſpectively to the ſquare on AB multiplied by the terms of the following ſeries: 6, 11, 21, 41, 81, 161, &c.

In the ſame way, if right lines be drawn from E, *e*, G, N, I, L, &c. there will ariſe numberleſs other ſeries. And if BC be taken equal to AB multiplied by any number, ſurd, fractional, or mixed, there will be obtained a great variety of ſeries, conſiſting reſpectively of terms, which are

are ſurd, fractional, or mixed. And by dividing BC, DE, *ge*, FG, MN, HI, KL, in different ways, according to pleaſure, we may apply the ſame method to fractional numbers, without altering the magnitude of BC. Thus, if BC be biſected, and a right line be drawn through the point of biſection parallel to AP, there will be found right lines, the ſquares on which are reſpectively equal to the ſquare on AB multiplied by a great number of fractions, having four for their common denominator, and ſo on.

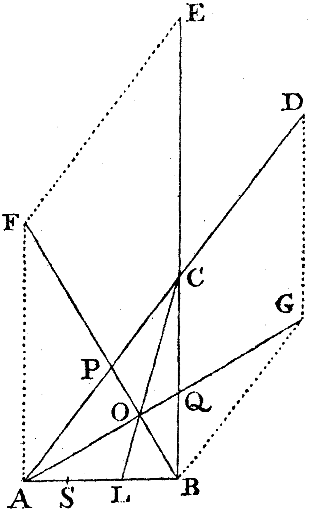
PROPOSITION IV. PROBLEM II.

To find a right line, the ſquare on which ſhall be equal to the ſquare on a given right line, divided by any number.

IF, uſing the figure of the immediately preceding problem, we ſuppoſe the given right line to be denoted by AL, the ſquares on AK, AH, AF, AD, AB, &c. will reſpectively be equal to the ſquare on AL multiplied by $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$, $\frac{1}{512}$, $\frac{1}{1024}$, &c. or divided by 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, &c.; and ſo on for other numbers, whole, ſurd, fractional, or mixed.

PROPOSITION V. PROBLEM III.

To cut off from a given right line a part expreſſed by any odd number.



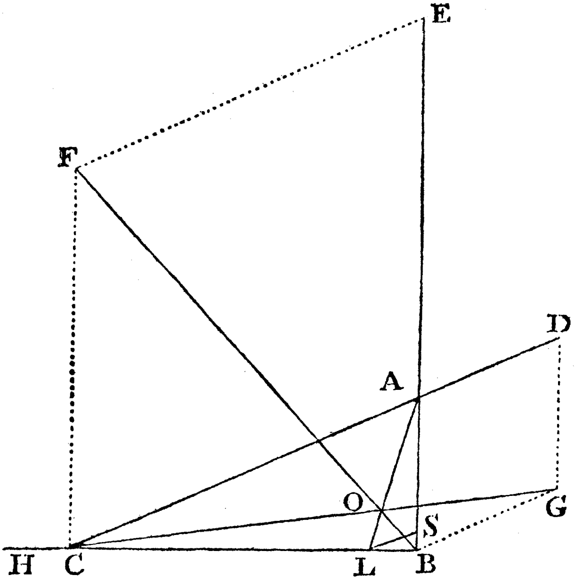
LET AB be the given right line. At right angles to it, at one of its extremities B, draw an indefinite right line BE. Let n be the given odd number, which expreſſes the part of AB which is to be cut off. Take BC ſuch a right line (prob. 8.), that the ſquare on it ſhall be equal to the ſquare on AB, multiplied by the number $\frac{n-1}{2}$. Draw CL as in the firſt theorem, and take LS equal to LB. I ſay, AS is that part of AB, which is expreſſed by the odd number n.

For the ſquare on AC being equal to the ſquares on AB, BC, is equal to the ſquare on AB multiplied by the number $\frac{n-1}{2} + 1$, or $\frac{n+1}{2}$. Therefore it appears (from prop. 1. and cor. 1. to 20. E. 6.), that AL is to LB as $\frac{n-1}{2} + 1$ to $\frac{n-1}{2}$. Conſequently, AS is equal to the part required. *Q. E. F.*

Thus, if the ſquare on BC be ſuppoſed ſucceſſively equal to the ſquare on AB multiplied by the terms of the ſeries 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, &c. the numbers of the ſeveral parts denoted by AS, will be 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, &c. which ſeries comprehends all odd numbers after 9, and might have begun from 3 had the other ſeries begun from 1.

PROPOSITION VI. PROBLEM IV.

To cut off from a given right line a part expreſſed by any even number.



Let m denote any even number in general. Draw any indefinite right line BH, and at right angles to it another BE. On BE ſet off the given right line BA, and from A, with the diſtance equal to a right line, the ſquare on which is equal to the ſquare on AB multiplied by the number

number $m-1$, interſect BH in ſome point C. From the vertex A of the triangle BAC draw AL as was directed in prop. 1. and draw LS parallel to CA. I ſay BL is ſuch a part of BC as is expreſſed by the number m; and that BS is the ſame part of AB. For it appears (from prop. 1.) that BL is to CL as 1 to $m-1$. Wherefore BL is the m^{th} part of BC. And BS is the ſame part of AB that BL is of BC (4. E. 6.)

Thus if the ſquare on AC be ſucceſſively denoted by the ſquare on AB multiplied by 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, &c. BS will be ſucceſſively ſuch a part of AB as is expreſſed by 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, &c.

PROPOSITION VII. THEOREM II.

If from the angles at the baſe of any right lined triangle, right lines be drawn to the alternate angles of rhombi *deſcribed on the other two ſides, and reciprocally applied to them produced, and through the interſection of theſe lines, a right line be drawn from the vertex to the baſe; the rectangle contained by the ſines of the angles at the extremities of one of the ſides, will be equal to the rectangle contained by the ſines of the angles at the extremities of the other* (b); *and the parallelepiped contained by the ſines of the*

(b) The author means, that $\overline{\text{ſin. ACL} \times \text{ſin. CAL}} = \text{ſin. BCL} \times \text{ſin. CBL}$ (ſee fig. prop. 1.). For ſin. ACL : ſin. L = AL : AC, and ſin. L : ſin. BCL = BC : BL. Take N, a third in proportion to AL, AC. Then, becauſe $AC^2 : BC^2 = AL : LB$, N will likewiſe *the angles of one of thoſe triangles, into which the original one is divided by the ſaid line drawn from the vertex. will be equal to the parallelepiped contained by the ſines of the angles of the other.*

COR. The two triangles, adjacent to the ſegments of the baſe, have to each other the proportion of the two adjacent to the ſides containing the vertical angle, or the proportion of the two into which the original triangle is divided; and any one of theſe pairs of triangles are as ſimilar figures deſcribed on the ſides, being as the ſegments of the baſe, which have to each other the duplicate proportion of the ſides.

PROPOSITION VIII. THEOREM III.

If from the angles at the hypotenuſe of any right angled right lined triangle, right lines be drawn to the alternate angles of ſquares deſcribed on the ſides containing the right angle, and from the point where the right line drawn from the right angle, through their interſection, meets the hypotenuſe, right lines be drawn to the points, where theſe lines meet the ſides; the lines ſo drawn will

make

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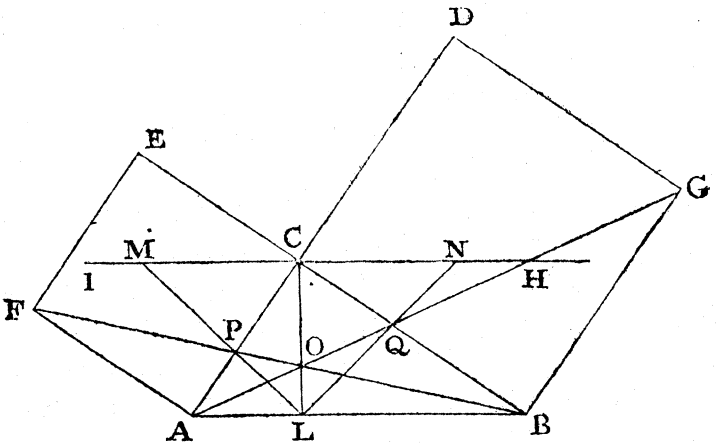
likewiſe be a third in proportion to LB, BC. Hence, ſin. ACL : ſin. L = AC : N, and ſin. L : ſin. BCL = N : BC. *Ex æquo perturbate* ſin. ACL : ſin. BCL = AC : BC. But ſin. CBL : ſin. CAL = AC : BC. Therefore, ſin. ACL : ſin. BCL = CBL : ſin. CAL, and $\overline{\text{ſin. ACL} \times \text{ſin. CAL}} = \overline{\text{ſin. BCL} \times \text{ſin. CBL}}$. *Q. E. D.* And hence, ſin. L × ſin. ACL × ſin. CAL = ſin. L × ſin. BCL × ſin. CBL, which is the ſecond branch of the propoſition. S. HORSLEY.

make equal angles with the hypotenuſe, and the right line drawn from the right angle to meet it; and will likewiſe have to each other the proportion of the ſides containing the right angle(c).

COR.

(c) For in the triangle ACB, having the angle at C right, conſtruct the figure as in the firſt propoſition, and join LP, LQ. The author affirms, that the angles



PLA, QLB, and alſo the angles CLP, QLC, are equal, and that PL : LQ = CA : CB. Produce LP, LQ, till they meet the right line IH in M, N. Now, becauſe MN, AB, are parallel, therefore AL : MC = AP : PC, and CN : LB = CQ (or PC) : QB. Therefore, AL × CN : MC × LB = AP : QB. But AP : QB = AL : LB (by theor. 1. cor. 5.). Therefore, AL × CN : MC × LB = AL : LB. Therefore, CN and CM are equal. But becauſe the angle ACB is right, CL is perpendicular to AB (by theor. 1. cor. 2.), and conſequently to MN. Therefore the angles MCL, NCL, are right, and MC being equal to CN, and CL common to the two triangles MCL, NCL, ML will be equal to LN, and the angles MLC, CML, equal to NLC, CNL, reſpectively. Hence it is evident, that the angles CLP, CLQ, are equal, and each is half a right angle; and likewiſe PLA, QLB, are equal, and each half a right angle.

Further,

COR. I. The alternate triangles of thoſe four, which have their vertices in the point, where the right line drawn from the right angle meets the hypotenuſe, are ſimilar, and have to each other the proportion of the ſegments of the hypotenuſe, or the duplicate proportion of the ſides containing the right angle.

COR. II. Either pair of the adjacent triangles lying on different ſides of the right line drawn from the right angle, and having their vertices in the interſection of the right lines drawn from the angles at the hypotenuſe, have to each other the proportion of the alternate triangles, having their vertices in the interſection of the firſt-mentioned line and the hypotenuſe.

COR. III. The trapezium or quadrilateral figure formed by the ſegments of the ſides adjacent to the right angle, and the right lines joining their extremities with the interſection of the hypotenuſe, and the right line drawn from the right angle to meet it, is capable of being inſcribed in a circle; and is divided at the interſection of right lines drawn from the angles at the hypotenuſe to the alternate angles of ſquares, deſcribed on the ſides containing the right angle, into triangles which are proportional to one another, and when taken two by two, as

they

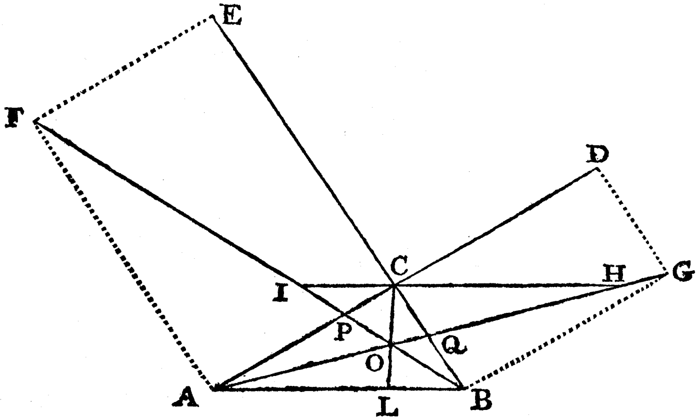
Further, the angles, PLA, CLQ, are equal, each being half a right angle. And the angles PAL, LCQ, are equal (by 8. Elem. 6.). Therefore the angles APL, CQL, will be equal, and the triangles APL, CQL, ſimilar, and the ſides ſubtending the equal angles proportional. Therefore, PL : LA = QL : LC. By permutation, LP : LQ = LA : LC. But LA : LC = AC : CB (by 8. Elem. 6.). Therefore, LP : LQ = AC : CB. *Q. E. D.* S. HORSLEY.

they lie adjacent on different ſides of the diagonal, are proportional to the unequal ſides of the trapezium, and to the two triangles into which the diagonal divides it. For OPL : OPC=LO : OC=OQL : OQC. Therefore, OPL : OQL =OPC : OQC=LPC : LQC=LP : LQ.

PROPOSITION IX. THEOREM IV.

If from the angles at the baſe of any right lined triangle, right lines be drawn to the alternate angles of rhomboids deſcribed on the other two ſides, and reciprocally applied to them produced, a right line drawn from the vertex through the interſection of theſe lines will cut the baſe into two parts, having to each other the proportion compounded of the proportion of the ſides, and of the proportion of the other two lines comprehending the rhomboids.

I SHALL ſubjoin a demonſtration of this theorem, ſince the firſt propoſition in theſe papers is only a particular caſe of it.



Let the triangle be ACB, the baſe AB, the rhomboids ACEF, CCDG; and let the right lines BF, AG, be drawn. Then, if from the vertex C through their interſection O, a right line COL be drawn to meet the baſe, the ſegments AL, LB, thereof will have to each other the proportion compounded of the proportions of AC to CB, and of CE to CD. For through the vertex C let a right line ICH be drawn parallel to AB, to meet BF, AG, produced, if neceſſary. Then, ſince the triangles CQH, CPI, are reſpectively equiangular to the triangles AQB, APB (15. and 29. E. 1.); the proportions of CH to AB, and of AB to IC are reſpectively equal to the proportions of CQ to QB, and of AP to PC (4. E. 6.). But the proportion of CH to IC is compounded of the proportions of CH to AB and of AB to IC; and conſequently is equal to a proportion compounded of the proportions of CQ to QB and of AP to PC. And ſince the triangles ACQ, APF, are reſpectively equiangular to the triangles BQG, BPC (15. and 29. E. 1.); the proportions of CQ to QB, and of AP to PC, are reſpectively equal to the proportions of AC to BG or CD, and of AE or CE to CB. Wherefore the proportion of CH to IC is equal to the proportion compounded of the proportions of AC to CD and of CE to CB, or of AC to CB and of CE to CD. But ſince the triangles COH, COI, are reſpectively equiangular to the triangles AOL, LOB, the proportion of CH to IC is equal to the proportion of AL to LB (4. E. 6.). Therefore the proportion of AL to LB is equal to the proportion compounded of the proportions of AC to CB and of CE to CD. *Q. E. D.*

SCHOLIUM. If CE, CD, be equal to each other, AL hath to LB the proportion of AC to CB, and CL biſects the angle ACB; if CE have to CD the inverſe proportion of AC to CB, AL is equal to LB; if CE have to CD the proportion of AC to CB, AL hath to LB the duplicate proportion of AC to CB; and univerſally, if CE have to CD any multiplicate proportion, n, of AC to CB, AL hath to LB ſuch a multiplicate proportion of AC to CB as is expreſſed by the number $n+1$. And if CE have to CD any multiplicate proportion m of CB to AC, AL will have to LB ſuch a multiplicate proportion of CB to AC, as is expreſſed by the number $m-1$.